BMATH FINAL EXAMINATION ALGEBRAIC GEOMETRY

Attempt all questions. Assume that the base field k, in all questions below, is algebraically closed. Total: 50 marks. Time: 3 hours

(1) Show that any nonsingular plane projective curve is irreducible. Is this true for an affine plane curve? Show that any irreducible plane projective curve has finitely many singular (or multiple) points. Is this true for a reducible plane projective curve? (10 marks)

(2) Classify, with proof, all discrete valuation rings R such that $k \subset R \subset Q(R) = k(t)$ (here t is a variable). Exhibit a bijective correspondence between such dvr's and the points of $\mathbb{P}^1_k$. (8 marks)

(3) Let $F \in k[x, y, z]$ be an irreducible homogenous polynomial of degree n, let $R = k[x, y, z]/(F)$ be the homogeneous coordinate ring of $V = V(F) \subset \mathbb{P}^2_k$. Let $R = \oplus R_d$, where R_d be the dth homogeneous component of R. Show that $\dim_k(R_d) = dn - \frac{n(n-3)}{2}$ for $d > n$. (8 marks)

(4) Verify the statement of Bezout's Theorem for the two curves $F(x, y, z) = y^2z - x(x - z)(x + z)$ and $G(x, y, z) = y^2z - x^3$, by computing both sides and showing they are equal. (8 marks)

(5) Prove that upto projective equivalence, there is only one irreducible conic in $\mathbb{P}^2$ (with equation $y^2 = xz$). (8 marks)

(6) Show that any irreducible cubic in $\mathbb{P}^2_k$ has at most one singular (or multiple) point, which is either a node or a cusp. (8 marks)